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CCS Research Report No. 651

Suggested Research Topics in Sensitivity and Stability Analysis for Semi-Infinite Programming Problems

by

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April 1990

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This research was partly supported by National Science Foundation Grant SES-8722504 with the Center for Cybernetic Studies, The University of Texas at Austin, and National Science Foundation Grant ECS-86-19859 and Office of Naval Research Grant N00014-89-J-1537 with the School of Engineering and Applied Science, George Washington University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

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SUGGESTED RESEARCH TOPICS IN SENSITIVITY AND STABILITY ANALYSIS FOR SEMI-INFINITE PROGRAMMING PROBLEMS

Abstract

We suggest several important research topics for semi-infinite programs whose problem functions and index sets contain parameters that are subject to perturbation. These include optimal value and optimal solution sensitivity and stability properties and penalty function approximation techniques. The approaches proposed are a natural carryover from parametric nonlinear programming, with emphasis on practical applicability and computability.

Key Words

Semi-infinite programming, perturbation analysis, sensitivity analysis, stability, parametric non-linear programming, penalty functions,

CONTENTS

SUGGESTED RESEARCH TOPICS IN SENSITIVITY AND STABILITY ANALYSIS FOR SEMI-INFINITE PROGRAMMING PROBLEMS

by

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1. INTRODUCTION

Our purpose is to suggest potentially fruitful areas of research in sensitivity and stability analysis for semi-infinite programming problems whose objective and constraint functions and index sets are perturbed by several parameters. The main subjects of this paper are the following:

I. Basic sensitivity and stability results: optimal value and optimal solution sensitivity and stability properties, i.e. continuity, differentiability, and lower and upper bounds of optimal values and optimal solutions of parametric semi-infinite programming problems;

II. Development of the penalty function method as a solution strategy and also as an algorithmic approximation technique for sensitivity information in semi-infinite programming;

III. Computational implementation.

The proposed areas of research are envisioned as extensions of the respective sensitivity and stability results and techniques for the standard nonlinear programming problem (NLP):

A. Basic sensitivity and stability characterizations, parameter derivatives and bounds.

B. Penalty function methodology.

C. Computer codes for solution, sensitivity and bound analysis for NLP, and demonstrations of practical applicability.

Throughout this paper, emphasis will be placed on computable results and computational aspects.

2. SEMI-INFINITE PROGRAMMING

2.1 Formulations and Applications

The semi-infinite programming problem (SIP) is a nonlinear programming problem formulated as follows:

$$\text{SIP:} \quad \min_{x} \; F(x)$$
$$\text{s.t.} \;\; G_i(x, t_i) \geqq 0 \quad \forall t_i \in T_i, \;\; i=1, \ldots, m,$$

where F: $E^n \to E^1$, G_i: $E^n \times E^{p_i} \to E^1$, $T_i \subset E^{p_i}$, and E^q is Euclidean q-space. The sets T_i are called index sets. We assume at least one T_i consists of infinitely many elements. The linear semi-infinite programming problem is a special case of

SIP and is formulated as:

$$\text{LSIP:} \quad \min_x \ c^T x$$
$$\text{s.t.} \quad a_i(t_i)^T x \geqq b_i(t_i) \quad \forall t_i \in T_i, \ i=1, \dots, m,$$

where c, $a_i(t_i) \in E^n$, $b_i(t_i) \in E^1$, and the superscript "T" denotes "transpose."

SIP arises in various kinds of system optimizations. Roughly speaking, optimization problems that are naturally formulated as SIP may be classified into three categories, according to the physical meanings of the index sets T_i.

(i) Decision problems under uncertainty: the case where $t_i \in T_i$ stands for the i-th "unknown" (i.e., the i-th opponent or i-th disturbance) variable, and the decision–maker wishes to optimize his objective F under the condition that the performance indexes G_i be kept over the permissible levels (zero), whatever the opponent's decision may be or whatever disturbances may happen. Examples: The problem of designing frame buildings [Polak (1983)], taking into account seismic force as a disturbance, and electronic circuit design problems [Polak (1983)] that include a production error parameter as the unknown factor. (These formulations by Polak also include time-dependent constraints, mentioned below.);

(ii) The case when the constraints depend on "time" or "space," i.e., $t_i \in T_i$ stands for time or space or frequency variables. Examples: An air pollution control problem [see Glashoff and Gustafson (1983)] which requires the minimum cost strategy to keep concentrations of a pollutant below the permissible level at <u>every</u> point in an area, and the design of

feedback control systems [Polak (1983)], over time or frequency domains;

(iii) Min-max (or Chebyshev or Uniform) approximation problems: Given a function f and a family $\mathcal{A} \equiv \{a(x, \cdot) \mid x \in X\}$ of approximation functions, the min-max approximation problem (MAP) on a domain T is formulated as follows:

$$\text{MAP:} \quad \min_{x \in X} \ \max_{t \in T} \ | \, a(x, t) - f(t) \, | \ .$$

This problem can be rewritten as an SIP:

$$\begin{aligned} \text{MAP:} \quad & \min_{(x,\sigma)} \quad \sigma \\ & \text{s.t.} \quad a(x, t) - f(t) + \sigma \geqq 0 \quad \forall t \in T \\ & \qquad\ \ - a(x, t) + f(t) + \sigma \geqq 0 \quad \forall t \in T \ . \end{aligned}$$

In particular, when $a(\cdot\, , t)$ *is linear in* x, MAP *is a special* case of LSIP. Most of the work for SIP and LSIP is motivated by MAP or its extensions.

2.2 Existing Results for SIP and LSIP

As for nonlinear programming (NLP), many aspects of SIP or LSIP have been studied, e.g.,

- Optimality conditions:

  - First order conditions [John (1948), Gehner (1974), Hettich and Jongen (1978)];
  - Second order conditions [Ben-Tal, Teboulle and Zowe (1979), Ioffe (1983), Shapiro (1985)];

- Duality theory [see papers in Fiacco and Kortanek (1983), Glashoff and Gustafson (1983), Anderson and Nash (1987)];
- Solution methods for LSIP:
  - Exchange algorithms, dual simplex methods and cutting plane methods [Hettich (1983), Glashoff and Gustafson (1983)];
  - Primal simplex methods [Anderson (1985), Anderson and Nash (1987)];
- Solution methods for SIP:
  - Cutting plane method [Blankenship and Falk (1976)];
  - Lagrange methods [Watson (1983, 1985), Coope and Watson (1985)];
  - Exact penalty method [Conn and Gould (1987)].

The difficulty of dealing with SIP or LSIP, of course, is the presence of an infinite numbers of constraints. A useful technique to overcome this difficulty replaces the infinitely many constraints with a finite number, without changing the optimal value [Borwein (1981, 1983), Jongen, Jonker and Twilt (1983)]. It is well known that this reduction is possible under mild assumptions (e.g., convexity). This fact forms a basis for a duality theory, and hence, for dual simplex methods. Furthermore, under stronger assumptions [e.g., assumptions under which the basic sensitivity theorem by Fiacco (1983, 1976) holds] for the inner problem,

$$IP_i(\overline{x}): \quad \min_{t_i \in T_i} G_i(\overline{x}, t_i),$$

of SIP at $\overline{x}$, we can obtain a finite number of functions t_{ij}, $j=1, \ldots, \sigma_i$, $i=1,\ldots, m$ which (locally) solve $IP_i(x)$ for $x \in N(\overline{x})$ (a neighborhood of $\overline{x}$), and then it can be

shown that SIP is (locally) equivalent to the following problem:

$$\text{RSIP:} \quad \min_x \quad F(x)$$
$$\text{s.t.} \quad G_i[x, t_{ij}(x)] \geqq 0, \quad j=1,\ldots,\sigma_i, \quad i=1, \ldots, m.$$

If $t_{ij} \in C^1$, the reduced problem RSIP is a standard NLP, and the theory for NLP will be applicable to RSIP. This idea is widely used in theoretical developments [Shapiro (1985), Zwier (1987)] and numerical solution methods [Hettich (1983), Watson (1983, 1985), Conn and Gould (1987)]. An extension of this technique should be useful in parametric SIP as well (see Section 3.2).

3. PROPOSED RESEARCH TOPICS

3.1 Parametric SIP

We are concerned with the question: If the data involved in an SIP changes, what happens to the optimal value or the optimal solutions? For example, let us consider the application problems mentioned in Section 2.1.

(i) Decision problems under uncertainty. If the decision-maker changes the permissible levels of the performance indices G_i, or if he assumes slightly different constraint sets T_i of the opponent or disturbance variables, how will the optimal decision change? Note that it is important to consider the latter case because, in most situations, the T_i are only "estimated" or "assumed" sets (e.g., the set of expected seismic forces).

(ii) Constraints depending on time or space coordination. How do changes in cost function, permissible levels, and area of air pollution control affect the optimal strategy? If we consider a different time interval on a

different frequency domain, what will happen to the control system?

(iii) Min-max approximation. If we change the function f and the domain T, slightly, what can we expect?

In order to answer these questions, we have to consider the parametric semi-infinite programming problem. Introducing the parameter vector $e \in E^q$ in SIP explicitly, we formulate this problem as follows:

$$\begin{array}{lll} \mathrm{SIP}(e): & \min\limits_{x} & F(x, e) \\ & \text{s.t.} & G_i(x, e, t_i) \geqq 0 \quad \forall t_i \in T_i(e), \quad i=1, \ldots, m . \end{array}$$

We define the constraint set, the optimal value, and the optimal solution set as follows:

$$\begin{aligned} R(e) &\equiv \{x \in E^n \mid G_i(x, e, t_i) \geqq 0 \quad \forall t_i \in T_i(e), \quad i=1, \ldots, m\}, \\ F^*(e) &\equiv \inf_{x \in R(e)} F(x, e), \\ S(e) &\equiv \{x^* \in R(e) \mid F(x^*, e) = F^*(e)\} . \end{aligned}$$

As briefly mentioned in Section 1, our proposed research includes:

I. Basic sensitivity and stability results for $F^*(e)$ and $S(e)$;

II. Penalty function approaches for SIP(e);

III. Emphasis on computability and computational implementation.

In the following three sections we shall discuss these subjects in more detail. First, we mention important relevant work in parametric semi-infinite programming. For standard (i,e., smooth and finitely constrained) parametric NLP, many useful results have been obtained. A survey of basic theoretical results, and the computable algorithmic approach to parametric NLP is given in the book by Fiacco

(1983). As for parametric SIP, however, there have not been as many results. In his book, Brosowski (1982) studies several types of perturbed SIP with fixed (i.e., nonperturbed) index sets, and investigates their "qualitative" properties (e.g., continuity of F^* and S^*, etc.). Nürnberger (1985) studies the (strong) unicity (i.e., uniqueness) of S(e) of problems similar to those studied by Brosowski. If, as in Brosowski's formulations, the index sets T_i are not perturbed, i.e., if the constraints in SIP(e) are of the following form:

$$G_i(x, e, t_i) \geqq 0 \quad \forall t_i \in T_i \, , \quad i=1, \ldots, m \, ,$$

then one can apply sensitivity and stability results of parametric infinite programming [Penot (1984), Lempio and Maurer (1980), da Silva (1985)], which treat perturbed constraints such as

$$G_i(x, e, \cdot\,) \in C_i \, , \quad i=1, \ldots, m \, ,$$

where C_i is a positive cone in a space of functions defined on T_i.

Another approach may be possible: defining minimal value functions of the inner problems of SIP(e) as

$$G_i^*(x, e) \equiv \inf_{t_i \in T_i(e)} G_i(x, e, t_i), \quad i=1, \ldots, m \, ,$$

we can rewrite SIP(e) as:

$$\begin{aligned} \text{SIP}^*(e): \quad & \min_x \quad F(x, e) \\ & \text{s.t.} \quad G_i^*(x, e) \geqq 0, \quad i=1, \ldots, m \, , \end{aligned}$$

which is of a standard parametric NLP form, with non–smooth constraint functions G_i^*. One can then apply sensitivity results from non–smooth parametric NLP [e.g., Auslender (1979), Rockafellar (1982)] to $SIP^*(e)$ to obtain estimates of the directional derivatives of F^* in terms of the generalized gradients [Clarke (1975)] of G_i^*. Recently, Zencke and Hettich (1987) obtained the directional derivative of F^* of a parametric linear SIP with fixed index sets, via duality arguments.

For min-max approximation problems, stability properties including the perturbation of the respective domains have been studied, e.g., Flachs (1985) and Dunham (1977) investigated the continuity of the best rational min-max approximation with respect to the domains.

Compared with past studies, our proposed research will have new and significant extensions in the following respects:

- Allowing the index sets to be perturbed;
- Emphasizing computability and computational aspects.

3.2 Basic Sensitivity and Stability Analysis for SIP(e)

Basic sensitivity and stability analysis for NLP includes the following important areas:

a. Local qualitative analysis, i.e., continuity of optimal values or optimal solutions set [see Section 2.2 in Fiacco (1983)];

b. Local quantitative analysis, i.e., estimation and characterization of (directional) derivatives of optimal values and optimal solutions [see Sections 2.3 and 2.4 in Fiacco (1983)];

c. Global qualitative analysis, i.e., convexity and concavity of optimal values

with respect to the parameter [see Fiacco and Kyparisis (1986)].

d. Global quantitative analysis, i.e., estimation of lower and upper bounds of optimal values and optimal solutions [see Chapter 9 in Fiacco (1983), Fiacco and Kyparisis (1988)]

We suggest further investigation of the analogous properties and measures for SIP(e). As mentioned in the previous section, viewing SIP(e) as a non-smooth parametric NLP, we can use general sensitivity theory to obtain an estimate of the directional derivative of F^*. Another theoretically and computationally useful approach will be the extension of the reduction idea mentioned in Section 2.2 to our SIP(e). Suppose that the inner problem:

$$IP_i(\overline{x}, \overline{e}): \quad \min_{t_i \epsilon T_i(\overline{e})} \; G_i(\overline{x}, \overline{e}, t_i)$$

at $(\overline{x}, \overline{e})$ satisfies the standard assumptions in Fiacco's basic sensitivity theorem [Fiacco (1983)], at <u>every</u> minimal solution t_i^*. Then there exists a finite number of perturbed local minimal solutions t_{ij}^* (x, e), $j=1, \ldots, \sigma_i$ of $IP_i(x, e)$ for $(x, e) \in N(\overline{x}, \overline{e})$, and SIP(e) is (locally) equivalent to the following reduced problem:

$$\begin{aligned} \text{RSIP(e):} \quad & \min_x \; F(x, e) \\ & \text{s.t.} \;\; G_i[x, e, t_{ij}(x, e)] \geqq 0, \;\; j=1, \ldots, \sigma_i, \;\; i=1, \ldots, m \,. \end{aligned}$$

Since, under the assumptions, t_{ij} are in C^1, we can apply existing smooth parametric NLP sensitivity and stability results to RSIP(e) to describe the local behavior of optimal values and optimal solutions of SIP(e).

As for global properties [i.e., c and d above] of SIP(e), (especially for optimal values), Fiacco's calculation technique for bounds on optimal values [Chap. 9 in Fiacco (1983)] will be applicable in a direct manner, since it is based on the convexity properties of the problem rather than on the finiteness or differentiability of the constraints.

3.3 Penalty Function Methods for SIP(e)

For standard parametric NLP, Fiacco (1983) has established a technique to estimate sensitivity information (e.g. a perturbed Karush-Kuhn-Tucker triple and its derivative) via penalty function algorithms [see also Armacost and Fiacco (1977, 1978)]. By this technique, one can obtain sensitivity information from trial points generated by a penalty function algorithm for solving a given NLP. Since it is easy to implement this technique in any penalty function algorithm, it provides a very useful sensitivity analysis tool in practice.

We suggest the adaptation of this technique to SIP(e). To this end, of course, a penalty function algorithm for solving SIP must be established. Unfortunately, however, it seems that no serious studies of penalty methods for SIP (or LSIP) have been made, except for Conn and Gould's recent exact penalty function method for SIP [Conn and Gould (1987)]. We propose to investigate application of classical penalty (especially, barrier) function methods to solve SIP problems. For example, denoting the penalty parameter by r,

$$P(x, r) \equiv F(x) + \sum_{i=1}^{m} \int_{t_i \epsilon T_i} \frac{r}{G_i(x, t_i)} \, dt_i \,,$$

$$P(x, r) \equiv F(x) - \sum_{i=1}^{m} \int_{t_i \epsilon T_i} r \, \ell n[G_i(x, t_i)] \, dt_i,$$

are simple extensions of classical barrier functions. It seems clear that a slight modification of finite-dimensional standard proofs of convergence will yield convergence theorems of methods for SIP based on these barrier function methods. Since the publication of Karmarkar (1984), many serious attempts to reconsider penalty function methods as solution methods for LP have been pursued [e.g., Gill, Murray, Saunders, Tomlin and Wright (1986); Megiddo (1986); Reneger (1988); Ben Daya and Shetty (1988)], and it has been reported that, at least for some particular classes of LPs, these "new" interior path methods are superior to the simplex method. In addition to the need to develop practical sensitivity analysis tools, this is another motivation for investigating the penalty function approach for solving SIP. Realizing that many existing algorithms for SIP (especially LSIP) can be regarded as (dual or primal) simplex methods [Hettich (1983), Glashoff and Gustafson (1983), Anderson and Nash (1987)], and that the effort to construct solution methods *(even for LSIP) continues, we are confident that* a *penalty function* approach will provide an efficient solution method as well as an important sensitivity analysis tool for SIP.

3.4 Computational Implementation

Based on the penalty function approach mentioned in the previous section, Armacost and Fiacco (1978) developed a computer code called SENSUMT to estimate the sensitivity information of a standard parametric NLP. As a preliminary experiment, we propose the use of SENSUMT to analyze SIP(e). Having solved the inner problem $IP_i(x, e)$ by an appropriate NLP algorithm, one may apply SENSUMT to the reduced problem RSIP(e) to obtain sensitivity information for SIP(e).

We also suggest investigation of the use of integral types of barrier functions like those mentioned in the previous section. Since we do not have experimental results for such barrier methods, their efficacy remains to be tested computationally.

3.5 Disclaimer

We do not presume to include all the vital issues and important current references in SIP, e.g., incisive work in the stability of the feasible set recently studied by Jonger, Twilt and Weber ["Semi-Infinite Optimization: Structure and Stability of the Feasible Set," Memo. No. 838, University of Twente, Enschede, The Netherlands, December 1989]. (See also, references to recent work in this paper.) The topics addressed are simply those of particular interest to the authors and deemed to be of considerable importance among those for which computable results may be obtained. Finally, we note that all the results addressed in this paper can be extended to include equality constraints. Since this extension is straightforward, we do not pursue the details here.

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